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Analytic Approximate Solution for a Nonlinear Oscillator Equation

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Abstract—Analytical approximate oscillatory solutions for the differential equation $u'' + \omega^2 u = \lambda u^m$ with the boundary conditions $u(0) = 0$, $u(1) = \alpha$ are derived using the Adomian decomposition method. The parameters ω, λ, and α are positive real numbers, and m is a positive integer.

The solutions are given in the form of series with easily computable terms and are valid for all ω including the critical frequencies. The validity of the solutions is verified through some numerical examples.

Keywords—Nonlinear oscillator, Decomposition method, Green's functions.

1. INTRODUCTION

In this paper we investigate the nonlinear oscillator equation

$$\frac{d^2u}{dx^2} + \omega^2 u = \lambda u^m, \tag{1.1}$$

where m is a positive integer and λ is a positive real number. Numerous techniques have been used to solve (1.1) subject to the initial conditions $u(0) = C$, $u'(0) = 0$, for the cases $m = 2$ and $m = 3$ known as Helmholtz and Duffing oscillators, respectively [1–4], and several perturbation methods are used to derive uniform asymptotic expansions for small λ [1,2]. Recently [5], a nonperturbative approximate solution is derived using the Adomian decomposition method [6] for general positive m.

Our concern in this work is the derivation of approximate analytical oscillatory solutions for the differential equation (1.1) associated with the boundary conditions

$$u(0) = 0, \quad u(1) = \alpha, \qquad \alpha > 0. \tag{1.2}$$

The problem now is more important due to the existence of critical values of ω for which regular perturbation is not valid. In addition, depending on the values of m, λ, ω and α, an oscillatory solution may or may not exist, and if it exists it may not be unique. For the special case $m = 2$ and $\omega = \pi$, an approximate solution is derived [7] using singular perturbation techniques.

In the present paper, we consider the boundary value problem (1.1) and (1.2) for positive integer m and using the decomposition method, we derive approximate analytical oscillatory solutions in the form of series with easily computable terms for all values of positive ω including the critical values $\omega = k\pi$. No linearization or smallness assumptions are needed in the method. In Section 2, we derive the necessary conditions for the existence of oscillatory solutions for (1.1). In Sections 3 and 4, the approximate solutions for $\omega \neq k\pi$ and $\omega = k\pi$, respectively, are derived. The validity of these solutions is verified through some numerical examples.

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2. PHASE-PLANE ANALYSIS

Some general features of equation (1.1) can be deduced from the phase diagram. The potential function $V(x)$ for this equation is defined by the formula

$$V'(x) = -\lambda u^m + \omega^2 u, \tag{2.1}$$

which leads to the equilibrium points: center at $u = 0$, two saddle points at $u = \pm(\omega^2/\lambda)^{1/(m-1)}$ if m is odd, and one saddle point at $u = (\omega^2/\lambda)^{1/(m-1)}$ if m is even. The phase paths are given by the energy equation

$$\frac{1}{2}\left(\frac{du}{dx}\right)^2 - \frac{\lambda u^{m+1}}{m+1} + \frac{\omega^2 u^2}{2} = \Gamma : \Gamma \text{ is constant}, \tag{2.2}$$

with the separatrix corresponding to

$$\Gamma_0 = \frac{\omega^2(m-1)}{2(m+1)}\left(\frac{\omega^2}{\lambda}\right)^{2/(m-1)}. \tag{2.3}$$

Therefore, we conclude that the motion is oscillatory if

$$\left|\dot{u}(0) < \omega\left(\frac{m-1}{m+1}\right)^{1/2}\left(\frac{\omega^2}{\lambda}\right)^{1/(m-1)}\right|, \tag{2.4}$$

which will be assumed to hold in the subsequent analysis.

3. METHOD OF SOLUTION

The decomposition method is well discussed in the literature; see the book of Adomian [8] for the latest development and applications of the method.

We first write equation (1.1) in the form

$$Lu = \lambda u^m, \tag{3.1}$$

where L denotes the linear operator

$$L \equiv \frac{d^2}{dx^2} + \omega^2. \tag{3.2}$$

We choose the linear operator to be (3.2) rather than $\frac{d^2}{dx^2}$ as it is usually done in this method, since we are interested in oscillatory solutions and these are generated by (3.2) more easily.

The operator L is invertible and its inverse is given by

$$L^{-1}[\cdot] = \int_0^1 g(x,s)[\cdot]\,ds, \tag{3.3}$$

where $g(x, s)$ is the Green's function which satisfies the boundary value problem [9]

$$Lg(x,s) = \delta(x-s), \tag{3.4}$$

$$g(0,s) = 0, \quad g(1,s) = 0. \tag{3.5}$$

Using the properties of Green's functions, we find that

$$g(x,s) = -\frac{1}{\omega\sin(\omega)}\begin{cases} \sin(\omega x)\sin(\omega(1-s)), & x < s, \\ \sin(\omega s)\sin(\omega(1-x)), & x > s. \end{cases} \tag{3.6}$$

Clearly, $g(x,s)$ is not defined for $\omega = k\pi$, which are known as the critical frequencies. In this section, we treat only the case $\omega \neq k\pi$ for which $g(x)$ is defined and unique. Applying L^{-1} on both sides of (3.1), and using the boundary conditions (1.2), we obtain

$$u = C\sin(\omega x) + \lambda \int_0^1 g(x,s)[u(s)]^m \, ds, \tag{3.7}$$

where C is an arbitrary constant to be determined. Now, we write u in the decomposition form

$$u = \sum_{n=0}^{\infty} u_n, \tag{3.8}$$

and expand the nonlinear term $[u(x)]^m$ as

$$[u(x)]^m = \sum_{n=0}^{\infty} A_n(x), \tag{3.9}$$

where the A_n are the Adomian polynomials [6] defined for $f(u)$ as

$$A_n = \sum_{\nu=0}^{n} C(\nu,n) f^{(\nu)}(u_0), \tag{3.10}$$

where the $C(\nu,n)$ are products, or sum of products of ν components of u whose subscript's sum equals to n with the result divided by the factorial of the number of repeated subscripts. In particular for $f(u) = u^m$, we have

$$A_0 = u_0^m, \tag{3.11}$$

$$A_1 = m u_1 u_0^{m-1}, \tag{3.12}$$

$$A_2 = m u_2 u_0^{m-1} + \frac{m(m-1)}{2!} u_1^2 u_0^{m-2}. \tag{3.13}$$

Next, we substitute (3.8) and (3.9) into (3.7) and define the components of u as follows:

$$u_0 = C\sin(\omega x), \tag{3.14}$$

$$u_n = \lambda \int_0^1 g(x,s) A_{n-1}(s) \, ds, \qquad n \geq 1. \tag{3.15}$$

Thus u_n can be determined recursively as far as we like, and the expression

$$\phi_n = \sum_{i=0}^{n-1} u_i \tag{3.16}$$

is the n-term approximation to u. The convergence of the decomposition method has been established in [10]. The constant C is determined using the boundary condition

$$\phi_n(1) = \alpha, \tag{3.17}$$

which gives

$$C = \frac{\alpha}{\sin(\omega)}. \tag{3.18}$$

In particular, using (3.11)–(3.15), a two term approximate solution is

$$\begin{aligned} \phi_2(x) = \frac{\alpha \sin(\omega x)}{\sin(\omega)} - \frac{\lambda \alpha^m}{\omega(\sin(\omega))^{m+1}} \Bigg[& I_m(x)\sin(\omega(1-x)) - \bar{I}_m(x)\cos(\omega)\sin(\omega x) \\ & + \frac{1}{\omega(m+1)} \left\{ (\sin(\omega))^{m+2}\sin(\omega x) - \sin(\omega)(\sin(\omega x))^{m+2} \right\} \Bigg], \end{aligned} \tag{3.19}$$

where

$$I_m(x) = \int_0^x (\sin(\omega s))^{m+1}\, ds, \tag{3.20}$$

$$\bar{I}_m(x) = \int_x^1 (\sin(\omega s))^{m+1}\, ds. \tag{3.21}$$

The integrals $I_m(x)$ and $\bar{I}_m(x)$ are given explicitly in [11, p. 131]. To verify the validity of the approximation (3.19), we consider a Duffing oscillator ($m = 3$), with the parameters $\omega^2 = 5/4$ and $\lambda = 1/2$ which has the exact solution [3]

$$u = \operatorname{sn}\left(x/\frac{1}{4}\right), \tag{3.22}$$

with $\alpha = \operatorname{sn}\big(1/(1/4)\big)$ where sn is the Jacobi elliptic function. Comparison of the values of the exact solution (3.22) and the approximate one (3.19) in Table 1 shows a good agreement, with maximum relative error 5.01×10^{-4}. Clearly, including more higher terms leads to highly accurate results.

Table 1. Comparison of the exact solution (3.22) and the two term approximation (3.19) for $m = 3$, $\omega^2 = 5/4$, $\lambda = 1/2$, $\alpha = \operatorname{sn}(x/(1/4))$.

x	u (Exact)	ϕ (Approx.)
0	0.000000	0.0000000
0.1	0.099792	0.0996758
0.2	0.198345	0.1981148
0.3	0.294466	0.294127
0.4	0.387042	0.386609
0.5	0.475083	0.474580
0.6	0.557734	0.557202
0.7	0.634293	0.633792
0.8	0.704212	0.703813
0.9	0.767085	0.766860
1.0	0.822636	0.822636

4. SOLUTIONS AT THE CRITICAL FREQUENCIES

Due to the nonexistence of Green's function $g(x,s)$ at $\omega = k\pi$, the solution (3.19) is not valid at these critical values. In fact, for these values of ω, the homogeneous boundary value problem

$$Lu = 0, \quad u(0) = 0, \quad u(1) = 0 \tag{4.1}$$

has the nontrivial solution $B\sin(k\pi x)$, and in this case we invert the operator L using the modified Green's function [9] $\bar{g}(x,s)$ which satisfies

$$L\bar{g}(x,s) = \delta(x-s) - 2\sin(k\pi x)\sin(k\pi s), \tag{4.2}$$

$$\bar{g}(0,s) = 0, \quad \bar{g}(1,s) = 0. \tag{4.3}$$

A solution of (4.2) and (4.3) is

$$\bar{g}(x,s) = \frac{1}{k\pi} x\left[\cos(k\pi x)\sin(k\pi x) + \sin(k\pi(x-s))H(x-s)\right], \tag{4.4}$$

where H is the Heaviside step function. The inversion of (1.1) using (1.2) gives

$$u = C'\sin(k\pi x) + \lambda \int_0^1 \bar{g}(x,s)[u(s)]^m\, ds + \alpha\left(\frac{\partial}{\partial s}\bar{g}(x,s)\right)_{s=1}, \tag{4.5}$$

which can be simplified into

$$u = C_1 \sin(k\pi x) + (-1)^k \alpha \cos(k\pi x) + \frac{\lambda}{k\pi} \int_x^1 \sin(k\pi(s-x))[u(s)]^m \, ds. \tag{4.6}$$

Now, to solve (4.6) we write u in the form

$$u = (-1)^k \alpha \cos(k\pi x) + \sum_{n=0}^{\infty} u_0, \qquad \text{where} \tag{4.7}$$

$$u_0 = C_1 \sin(k\pi x), \qquad \text{and} \tag{4.8}$$

$$u_n = \frac{\lambda}{k\pi} \int_x^1 [\sin(k\pi(s-x))] A_{n-1}(s) \, ds, \qquad n \geq 1. \tag{4.9}$$

In particular, a three term approximate solution is

$$\begin{aligned} \phi_3 = {} & C_1 \sin(k\pi x) + (-1)^k \alpha \cos(k\pi x) + \frac{\lambda C_1^m}{k\pi} \psi(x) \\ & + \frac{m\lambda^2 C_1^{2m-1}}{k^2\pi^2} \int_x^1 \psi(s)(\sin(k\pi s))^{m-1} \sin(k\pi(s-x)) \, ds, \end{aligned} \tag{4.10}$$

where

$$\psi(x) = \bar{I}_m(x) \cos(k\pi x) + \frac{(\sin(k\pi x))^{m+2}}{k\pi(m+1)}. \tag{4.11}$$

The boundary condition at $x = 1$ is satisfied while the condition at $x = 0$ gives the following equation for C_1:

$$(-1)^k \alpha + \frac{\lambda C_1^m}{k\pi} \bar{I}_m(0) + \frac{m\lambda^2 C_1^{2m-1}}{k^2\pi^2} J_m(0) = 0, \tag{4.12}$$

where

$$J_m(0) = \int_0^1 \psi(s)(\sin(k\pi s))^m \, ds. \tag{4.13}$$

Using formulae from [11, p. 369] we have

$$\bar{I}_m(0) = \begin{cases} \dfrac{1}{2^{m+1}} \dbinom{m+1}{\frac{m+1}{2}}, & m \text{ odd}, \ k \text{ arbitrary}, \\ \dfrac{2^{m+1}}{k\pi(m+1)} \dbinom{m}{\frac{m}{2}}^{-1}, & m \text{ even}, \ k \text{ odd}, \\ 0, & m \text{ even}, \ k \text{ even}, \end{cases} \tag{4.14}$$

and after some computations we find that

$$J_m(0) = N_m \begin{cases} \dbinom{2m+2}{m+1} + 2\sum_{j=0}^{m/2} \dbinom{m+1}{j}^2, & m \text{ even}, \\ \dbinom{2m+2}{m+1} + \dbinom{m+2}{\frac{m+1}{2}}^2 + 2\sum_{j=0}^{(m-1)/2} \dbinom{m+1}{j}^2, & m \text{ odd}, \end{cases} \tag{4.15}$$

where

$$N_m = \frac{1}{k\pi(m+1)2^{2m+2}}, \tag{4.16}$$

and this completes the determination of $\phi_3(x)$ for all the critical frequencies.

For comparison purposes, we consider the Helmholtz oscillator ($m = 2$). Using the shooting method, we obtained the numerical solutions when $\lambda = 0.01$ and $\alpha = 1$ for different values of k. The results for $k = 3$ and 4 are listed in Table 2 and Table 3, along with the values computed from

Table 2. Comparison of the numerical solutions for $m = 2$, $\omega = 3\pi$, $\lambda = 0.01$, $\alpha = 1$ and the three-term approximation (4.10).

(x)	Numerical Solutions (u)		Approximate Solutions (ϕ_3)	
	First Sol.	Second Sol.	First Sol.	Second Sol.
0	0.0000000	0.0000000	0.0000000	0.0000000
0.1	6.520725E+1	−6.691268E+1	6.519428E+1	−6.689762E+1
0.2	7.702532E+1	−7.827259E+1	7.703509E+1	−7.828197E+1
0.3	2.582453E+1	−2.460362E+1	2.584368E+1	−2.462396E+1
0.4	−4.653903E+1	4.947730E+1	−4.652957E+1	4.946615E+1
0.5	−8.030622E+1	8.302015E+1	−8.030883E+1	8.302156E+1
0.6	−4.735091E+1	4.867126E+1	−4.735963E+1	4.867996E+1
0.7	2.487523E+1	−2.555652E+1	2.486788E+1	−2.554819E+1
0.8	7.671798E+1	−7.858353E+1	7.671803E+1	−7.858227E+1
0.9	6.579207E+1	−6.632205E+1	6.579736E+1	−6.632642E+1
1.0	1.000000E+0	1.000000E+0	1.000000E+0	1.000000E+0

Table 3. Comparison of the numerical solutions for $m = 2$, $\omega = 4\pi$, $\lambda = 0.01$, $\alpha = 1$ and the three-term approximation (4.10).

x	Numerical Sol. (u)	Approximate Sol. (ϕ_3)
0.0	0.0000000	0.0000000
0.1	−3.432510E+2	−3.434145E+2
0.2	−2.040093E+2	−2.041039E+2
0.3	2.217509E+2	2.218518E+2
0.4	3.460561E+2	3.462174E+2
0.5	5.000000E−1	5.000000E−1
0.6	−3.431011E+2	−3.432599E+2
0.7	−2.044214E+2	−2.045084E+2
0.8	2.213542E+2	2.214473E+2
0.9	3.462147E+2	3.463719E+2
1.0	1.000000E+0	1.000000E+0

the present approximate solutions (4.10). The results show a good agreement, with maximum relative error of 4.7×10^{-4}.

A final important note worth mentioning is that when we used the shooting method to find a numerical solution, a problem of convergence arose due to the high sensitivity of the method to the initial guess of $u'(1)$. This problem is handled by using the derivatives of the present approximate solution as an initial guess, and this leads to a fast convergence of the numerical scheme. There is still a question to be resolved concerning the number of solutions of the boundary value problem (1.1) and (1.2) in the case of critical frequencies.

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